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Lecture 8 Addendum: Confidence Interval for σ^2

Recall that if

$$X \sim N(\mu, \sigma^2)$$

then

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

and

$$Z^2 \sim \chi_1^2 \equiv \Gamma(\frac{1}{2}, \frac{1}{2})$$

by **Example 4.3**.

If $x_1, x_2, ..., x_n$ are i.i.d. $N(\mu, \sigma^2)$ then

$$S_n^2 = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} = \sum_{i=1}^{n} Z_i^2 \sim \Gamma(\frac{n}{2}, \frac{1}{2}) \equiv \chi_n^2$$

by **Remark 6.10** applied to a sum of n independent $\Gamma(\frac{1}{2}, \frac{1}{2})$ random variables. We can write S_n^2 as

$$S_n^2 = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} [(x_i - \bar{x}) + (\bar{x} - \mu)]^2$$

$$= \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} + \left(\frac{n^{\frac{1}{2}} (\bar{x} - \mu)}{\sigma} \right)^2$$

because $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$. We will exploit the Pythagorean Relation when we discuss the linear model in **Lectures 14**, **15** and **16**.

Hence,

$$S_n^2 = U + Z^2$$

and because $\bar{x}$ and $\sum_{i=1}^{n}(x_i - \bar{x})^2$ are independent by Homework Assignment 7, $S_n^2 \sim \Gamma(\frac{n}{2}, \frac{1}{2})$ and $Z^2 \sim \Gamma(\frac{1}{2}, \frac{1}{2})$. It follows also by Homework Assignment 7 that

$$U \sim \Gamma(\frac{n-1}{2}, \frac{1}{2}) \equiv \chi^2_{n-1}$$

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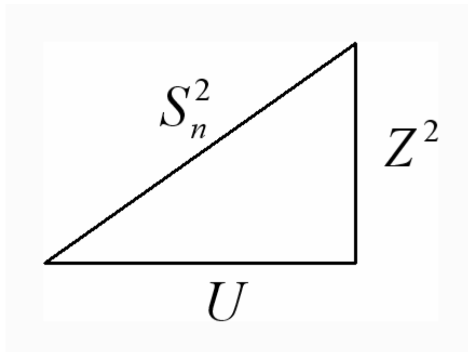

Figure 8A.1 The Pythagorean Relation Between U, Z^2 and S_n^2.

Now

$$\hat{\sigma}^2 = n^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

and

$$U = \frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}.$$

We have (Figure 8.A.2)

$$\Pr(\chi^2_{n-1}(0.025) \le \frac{n\hat{\sigma}^2}{\sigma^2} \le \chi^2_{n-1}(0.975)) = 0.95$$

or

$$\Pr(\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(0.975)} \le \sigma^2 \le \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(0.025)}) = 0.95$$

is an exact 95% confidence interval for σ^2, where $\chi^2_{n-1}(\alpha)$ is the α^{th} quantile of the χ^2-distribution with $n-1$ degrees of freedom (Figure 8.A.2). This confidence interval is not symmetric (Fig. 8.A.2). Our definition of $\chi^2_{n-1}(\alpha)$ is $\chi^2_{n-1}(1-\alpha)$ in Rice, p. 280.

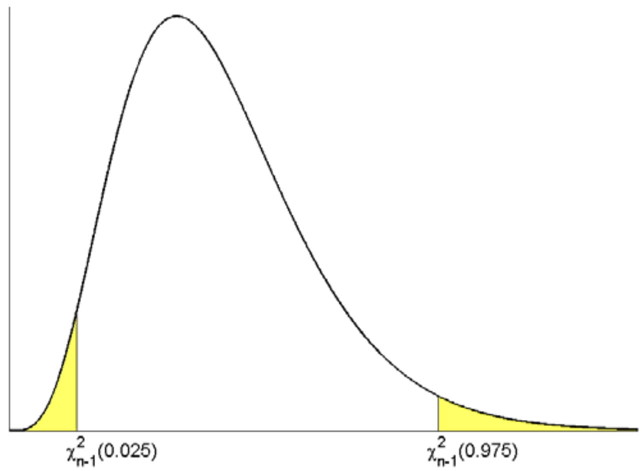

Figure 8.A.2. Limits for a $1-\alpha$ confidence interval based on a χ^2-distribution with $n-1$ degrees of freedoms. Notice that the confidence interval is not symmetric.